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On Hermite interpolation by Cauchy–Vandermonde systems: the Lagrange formula, the adjoint and the inverse of a Cauchy–Vandermonde matrix

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Abstract

For a given Cauchy–Vandermonde system and for given multiple nodes a Lagrange-type formula for the interpolant is derived, interpolating a given function in the sense of Hermite. We give explicit analytic representations of the basic functions in terms of the nodes and prescribed poles. They are used to derive formulas for the entries of the adjoint of the confluent Cauchy–Vandermonde matrix corresponding to the interpolation problem thus providing an explicit representation of its inverse.

Keywords: Cauchy–Vandermonde systems; Hermite interpolation; Inverse of a Cauchy–Vandermonde matrix

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1. Cauchy–Vandermonde systems

Suppose we are given a sequence $\mathscr{B} = (b_1, b_2, \ldots)$ of not necessarily distinct points of the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. With $\mathscr{B}$ we associate a system $\mathscr{U} = (u_1, u_2, \ldots)$ of basic rational functions defined by

$$u_j(z) = \begin{cases} z^{\nu_j(b_j)} & \text{if } b_j = \infty, \\ (z - b_j)^{-\nu_j(b_j) - 1} & \text{if } b_j \in \mathbb{C}. \end{cases} \tag{1}$$

Here $\nu_j(b)$ denotes the multiplicity of b in the sequence $(b_1, \ldots, b_{j-1})$. The system $\mathscr{U}$ will be called the *Cauchy–Vandermonde system* associated with the pole sequence $\mathscr{B}$. For any $k \in \mathbb{N}$ fixed with the initial section of $\mathscr{B}$

$$\mathscr{B}_k := (b_1, \ldots, b_k) \tag{2}$$

there corresponds the basis

$$\mathscr{U}_k := (u_1, \ldots, u_k) \tag{3}$$

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of the k-dimensional *Cauchy–Vandermonde space* span $\mathscr{U}_k$. It is well known [3, 4] that for every k, $\mathscr{U}_k$ is an extended complete Čebyšev system on $\mathbb{C}\setminus\{b_1,\ldots,b_k\}$, in particular for $k\in\mathbb{N}$ fixed and for any given system

$$\mathscr{A}_k := (a_1,\ldots,a_k) \tag{4}$$

of not necessarily distinct complex numbers a_i and for any complex function f which is defined and sufficiently smooth at the nodes a_i there is a unique element $u\in$ span $\mathscr{U}_k$ satisfying the interpolation conditions

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu_i(a_i)}(u-f)(a_i)=0 \quad (i=1,\ldots,k). \tag{5}$$

Here $\mu_i(a)$ denotes the multiplicity of a in the sequence $(a_1,\ldots,a_{i-1})$. In [2, 3] a Neville–Aitken formula and in [5] a Newton formula computing the interpolant u recursively are given. The aim of this note is twofold. First we are going to derive a Lagrange-type formula

$$u(z)=\sum_{i=1}^{k} l_i(z)\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu_i(a_i)} f(a_i). \tag{6}$$

More precisely, we will derive explicit representations of the basic Lagrange functions l_i in terms of the nodes (4) and the poles (2) involved. Of course, if all poles are prescribed to be at infinity this result must contain the well-known Lagrange–Hermite formula for interpolation by algebraic polynomials which can be found in [1]. In a second step we will use the representations of the basic rational functions l_i to compute the entries of the inverse of the Cauchy–Vandermonde matrix

$$V := (\langle L_i, u_j\rangle)_{i=1,\ldots,k}^{j=1,\ldots,k}, \tag{7}$$

where by L_i we denote the Hermite functionals

$$f \mapsto \langle L_i, f\rangle = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu_i(a_i)} f(a_i). \tag{8}$$

2. The Lagrange formula

The basic Lagrange functions $l_j\,(j=1,\ldots,k)$ for the k-dimensional Cauchy–Vandermonde space span $\mathscr{U}_k$ are uniquely determined by the conditions of biorthogonality

$$\langle L_i, l_j\rangle = \delta_{i,j} \quad (i,j=1,\ldots,k).$$

Evidently,

$$l_j = \frac{1}{\det V}\begin{vmatrix} \langle L_1, u_1\rangle & \cdots & \langle L_1, u_k\rangle \\ \vdots & & \vdots \\ \langle L_{j-1}, u_1\rangle & \cdots & \langle L_{j-1}, u_k\rangle \\ u_1 & \cdots & u_k \\ \langle L_{j+1}, u_1\rangle & \cdots & \langle L_{j+1}, u_k\rangle \\ \vdots & & \vdots \\ \langle L_k, u_1\rangle & \cdots & \langle L_k, u_k\rangle \end{vmatrix}, \tag{9}$$

where V is the confluent Cauchy–Vandermonde matrix (7) and where the numerator determinant is defined by its formal Laplacian expansion along its jth row. Obviously, knowing the coefficients $c_{j,t}$ of the expansion

$$l_j = \sum_{t=1}^{k} c_{j,t} \cdot u_t \quad (j = 1, \ldots, k), \tag{10}$$

this yields explicit representations of the adjoint and of the inverse of V. In fact, the adjoint V_{adj} of V equals $\det V \cdot C^{\mathrm{T}}$, where $C := (c_{j,t})_{j=1,\ldots,k}^{t=1,\ldots,k}$, hence

$$V^{-1} = C^{\mathrm{T}}. \tag{11}$$

There is an explicit formula for $\det V$ provided the nodes and poles are *consistently ordered*, i.e.

$$\mathscr{A}_k = (a_1, \ldots, a_k) = (\underbrace{\alpha_1, \ldots, \alpha_1}_{m_1}, \alpha_2, \ldots, \alpha_{p-1}, \underbrace{\alpha_p, \ldots, \alpha_p}_{m_p}), \tag{12}$$

$$\mathscr{B}_k = (b_1, \ldots, b_k) = (\underbrace{\beta_1, \ldots, \beta_1}_{n_1}, \beta_2, \ldots, \beta_{q-1}, \underbrace{\beta_q, \ldots, \beta_q}_{n_q}), \tag{13}$$

where $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$ are pairwise distinct and $m_1 + \cdots + m_p = k$ and $n_1 + \cdots + n_q = k$. Under these assumptions the Cauchy–Vandermonde matrix (7) has the determinant

$$\det V = \operatorname{mult}(\mathscr{A}_k) \cdot \frac{\prod^{*k}_{\substack{i,j=1 \\ i>j}} (a_i - a_j) \prod^{*k}_{\substack{i,j=1 \\ i>j}} (b_i - b_j)}{\prod^{k}_{\substack{i,j=1 \\ i\geqslant j}} (a_i - b_j) \prod^{*k}_{\substack{i,j=1 \\ i>j}} (b_i - a_j)}, \tag{14}$$

which was derived in [3] (cf. also [2, 4]). Here we use the notations

$$\operatorname{mult}(\mathscr{A}_k) = \prod_{i=1}^{k} \mu_i(a_i)!$$

and for a finite index set J and elements $\gamma_j \in \bar{\mathbb{C}}$

$$\prod_{j\in J}^{*} \gamma_j := \prod_{j\in J} \gamma_j^{*},$$

with

$$\gamma_j^* := \begin{cases} 1 & \text{iff } \gamma_j = \infty \text{ or } \gamma_j = 0, \\ \gamma_j & \text{iff } \gamma_j \in \mathbb{C}\setminus\{0\}. \end{cases}$$

Of course, there is no loss of generality in assuming that the nodes and poles are ordered consistently. This only means reordering the system $\mathscr{U}_k$ keeping it to be an extended complete Čebyšev system on $\mathbb{C}\setminus\{b_1,\dots,b_k\}$ and reordering the sum (6) according to the permutation of $\mathscr{A}_k$ to get the node system consistently ordered. Assuming consistently ordered systems (12) and (13) leads to a simple sign factor in the determinant (14).

In order to derive an explicit representation of the inverse of V it is important that we can change easily between the one-index enumerations of the Hermite functionals (8) corresponding to the one-index enumeration of the nodes (4) and the two-index enumeration

$$\langle L_i, f\rangle = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\rho} f(\alpha_r) \quad (r = 1,\dots,p;\ \rho = 0,\dots,m_r - 1). \tag{15}$$

This is done by the one-to-one correspondence

$$(r,\rho) \overset{\varphi}{\mapsto} i = \varphi(r,\rho) := m_1 + \cdots + m_{r-1} + \rho + 1 \tag{16}$$

shown in the table

(r,ρ)	$(1,0)$	$(1,1)$	$\dots$	$(1,m_1-1)$	$(2,0)$	$\dots$	$(2,m_2-1)$	$\dots$	(r,ρ)	$\dots$	(p,m_p-1)
$i=\varphi(r,\rho)$	1	2	$\dots$	m_1	m_1+1	$\dots$	m_1+m_2	$\dots$	$m_1+\cdots+m_{r-1}+\rho+1$	$\dots$	k

Similarly, the two enumerations of the Cauchy–Vandermonde functions (3) corresponding to the one-index enumeration (2) of the poles and the two-index enumeration

$$u_j = u_{m,\mu} \quad (m = 1,\dots,q;\ \mu = 1,\dots,n_m),$$

where

$$u_{m,\mu}(z) := \begin{cases} \dfrac{1}{(z-\beta_m)^{\mu}}, & m = 1,\dots,q-1;\ \mu = 1,\dots,n_m, \\ z^{\mu-1}, & m = q;\ \mu = 1,\dots,n_q \end{cases} \tag{17}$$

is realized by the one-to-one mapping

$$(m,\mu) \overset{\psi}{\mapsto} j = \psi(m,\mu) = n_1 + \cdots + n_{m-1} + \mu. \tag{18}$$

Observe that for convenience we assume the pole ∞ to be represented by β_q.

In order to derive a Lagrange-type formula (6) we need some notation:

$$Q(z) := \prod_{j=1}^{k}{}^{*} (z - b_j) = \prod_{t=1}^{q-1} (z - \beta_t)^{n_t},$$

$$\omega_l(z) := \prod_{s=1,\, s \neq l}^{p} (z - \alpha_s)^{m_s} \quad (l = 1, \ldots, p),$$

$$v_{l,\lambda}(z) := \frac{1}{\lambda!}(z - \alpha_l)^{\lambda} \quad (l = 1, \ldots, p;\ \lambda = 0, \ldots, m_l - 1),$$

$$P_{l,\lambda}(z) := \sum_{i=0}^{m_l - \lambda - 1} \frac{1}{i!} d_l^i \left(\frac{Q}{\omega_l}\right)(z - \alpha_l)^i$$

= Taylor's polynomial of order $m_l - \lambda - 1$ of the function Q/ω_l developed at the point $z = \alpha_l$,

$$d_l^i(\cdot) := \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^i (\cdot)_{z = \alpha_l} \quad (l = 1, \ldots, p;\ i = 0, \ldots, m_l - 1). \tag{19}$$

Theorem 1. *Assume that the node system* (3) *and the pole system* (2) *are disjoint. Suppose that these systems when consistently ordered are identical with* (12) *and* (13), *respectively. Then the Lagrange-type basis functions* (9) *are*

$$l_{\varphi(l,\lambda)}(z) := \omega_l^{\lambda}(z) := \frac{\omega_l(z)}{Q(z)} P_{l,\lambda}(z) \cdot v_{l,\lambda}(z) \quad (l = 1, \ldots, p;\ \lambda = 0, \ldots, m_l - 1). \tag{20}$$

The functions $\omega_l^{\lambda} \in \operatorname{span} \mathscr{U}_k$ *are uniquely determined by the biorthogonality relations*

$$d_s^{\sigma} \omega_l^{\lambda} = \delta_{(s,\sigma),(l,\lambda)} = \delta_{s,l} \cdot \delta_{\sigma,\lambda} \ (s = 1, \ldots, p;\ \sigma = 0, \ldots, m_s - 1), (l = 1, \ldots, p;\ \lambda = 0, \ldots, m_l - 1). \tag{21}$$

Proof. By partial fraction decomposition first we observe that $\omega_l^{\lambda} \in \operatorname{span} \mathscr{U}_k$ for all $l = 1, \ldots, p$ and $\lambda = 0, \ldots, m_l - 1$. Moreover, if $s \neq l$ then in view of Leibniz' rule $d_s^{\sigma} \omega_l^{\lambda} = 0$ since then ω_l^{λ} contains the factor $(z - \alpha_s)^{m_s}$. Supposing now $s = l$ we must show that

$$d_l^{\sigma} \omega_l^{\lambda} = \delta_{\sigma,\lambda} \quad (\sigma, \lambda = 0, \ldots, m_l - 1).$$

Again, by Leibniz' rule this is clear for $\sigma < \lambda$. When $\sigma \geqslant \lambda$ this is equivalent with

$$d_l^{\sigma}(u \cdot v) = \delta_{\sigma,\lambda} \quad (\lambda \leqslant \sigma \leqslant m_l - 1),$$

where we have set $u := \omega_l \cdot P_{l,\lambda}/Q$ and $v := v_{l,\lambda}$ and d_l^i is defined by (19). Using Leibniz' rule repeatedly we find

$$d_l^{\sigma}(u \cdot v) = \binom{\sigma}{\lambda} d_l^{\sigma - \lambda} u = \binom{\sigma}{\lambda} \cdot \sum_{\rho=0}^{\sigma - \lambda} \binom{\sigma - \lambda}{\rho} d_l^{\rho}\left(\frac{\omega_l}{Q}\right) d_l^{\sigma - \lambda - \rho} P_{l,\lambda}.$$

But it is easily seen that $d_l^{\sigma-\lambda-\rho}P_{l,\lambda} = d_l^{\sigma-\lambda-\rho}(Q/\omega_l)$ for $0 \leqslant \sigma - \lambda - \rho \leqslant m_l - \lambda - 1$. Therefore,

$$d_l^{\sigma}(u \cdot v) = \binom{\sigma}{\lambda} d_l^{\sigma-\lambda}\left(\frac{\omega_l}{Q} \cdot \frac{Q}{\omega_l}\right) = \delta_{\sigma,\lambda}. \qquad \square$$

In view of a well-known theorem of linear algebra as an immediate consequence of the biorthogonality relations (21) we have the following corollary.

Corollary. *Under the assumptions of Theorem* 1 *there holds the Lagrange–Hermite interpolation formula*

$$u(z) = \sum_{l=1}^{p} \sum_{\lambda=0}^{m_l-1} \omega_l^{\lambda}(z) \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\lambda} f(\alpha_l),$$

which is identical with (6) *after renumbering the basic Lagrange functions and the Hermite functionals according to* φ [*cf.* (16)].

3. Partial fraction decomposition of the Lagrange basis functions

The aim of this section is to compute the coefficients $A_{m,u}^{l,\lambda}$ of the partial fraction decomposition

$$\omega_l^{\lambda} = \sum_{m=1}^{q} \sum_{\mu=1}^{n_m} A_{m,\mu}^{l,\lambda} \cdot u_{m,\mu}, \tag{22}$$

where ω_l^{λ} is given in Theorem 1 and $u_{m,\mu}$ is defined by (17). We will give explicit formulas for the $A_{m,\mu}^{l,\lambda}$ in terms of the nodes and poles involved.

Multiplying (22) by $Q(z)$ we see that this equation is equivalent with

$$\begin{aligned}\omega_l(z) \cdot P_{l,\lambda}(z) \cdot v_{l,\lambda}(z) = &\sum_{m=1}^{q-1} \sum_{\mu=1}^{n_m} A_{m,\mu}^{l,\lambda}\left(\prod_{t=1,t\neq m}^{q-1} (z-\beta_t)^{n_t}\right)(z-\beta_m)^{n_m-\mu} \\ &+ Q(z) \sum_{\mu=1}^{n_q} A_{q,\mu}^{l,\lambda} \cdot z^{\mu-1}.\end{aligned} \tag{23}$$

We shall make use of the shorthand notations

$$D_m^{\tau}(\cdot) := \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} (\cdot)_{z=\beta_m} \quad (m = 1, \ldots, q; \tau = 1, \ldots, n_m).$$

Theorem 2. *Under the assumptions of Theorem* 1 *we have for every* $l = 1, \ldots, p$; $\lambda = 0, \ldots, m_l - 1$

$$A_{m,\mu}^{l,\lambda} = \begin{cases} \dfrac{D_m^{n_m-\mu}[\omega_l \cdot P_{l,\lambda} \cdot v_{l,\lambda}]}{(n_m-\mu)! \prod_{t=1,t\neq m}^{q-1} (\beta_m - \beta_t)^{n_t}} & \text{if } m = 1, \ldots, q-1; \mu = 1, \ldots, n_m, \quad (24\text{a}) \\[2ex] \dfrac{D_q^{n_q-\mu}}{(n_q-\mu)!}\left[\dfrac{\omega_l(z) P_{l,\lambda}(z) \cdot v_{l,\lambda}(z)}{Q(z) \cdot z^{n_q-1}}\right] & \text{if } m = q; \mu = 1, \ldots, n_q. \quad (24\text{b}) \end{cases}$$

Proof. Formula (24a) results immediately by applying $D_m^{n_m-\mu}/(n_m-\mu)!$ to both sides of (23) in view of Leibniz' rule. Similarly, (24b) results by applying $D_q^{n_q-\mu}/(n_q-\mu)!$ to both sides of the equation obtained by dividing (22) by z^{n_q-1}. □

In order to express the coefficients $A_{m,\mu}^{l,\lambda}$ in terms of the nodes and poles explicitly we have to carry out some elementary but tedious calculations. To begin with we compute first the coefficients of the polynomial $P_{l,\lambda}$. Once more we use Leibniz' formula

$$d_l^i\left(\frac{Q}{\omega_l}\right)=\sum_{j=0}^{i}\binom{i}{j}d_l^j Q\cdot d_l^{i-j}\left(\frac{1}{\omega_l}\right)$$

and express $d_l^i Q$ and $d_l^h(1/\omega_l)$ separately. Let us introduce the shorthand notations

$$Z_l:=z-\alpha_l,\qquad Y_{t,l}:=\beta_t-\alpha_l,\qquad Z_{s,l}:=\alpha_s-\alpha_l.$$

Then

$$Q(z)=\prod_{t=1}^{q-1}(z-\beta_t)^{n_t}=\prod_{t=1}^{q-1}(Z_l-Y_{t,l})^{n_t}=\prod_{t=1}^{q-1}\sum_{\tau_t=0}^{n_t}\binom{n_t}{\tau_t}Z_l^{\tau_t}(-Y_{t,l})^{n_t-\tau_t}=\sum_{j=0}^{n_1+\cdots+n_{q-1}}\frac{Z_l^j}{j!}\cdot d_l^j Q,$$

with

$$d_l^j Q=j!\sum_{\substack{(\tau_1,\ldots,\tau_{q-1})\in\mathbb{N}_0^{q-1}\\ \tau_1+\cdots+\tau_{q-1}=j}}\binom{n_1}{\tau_1}\cdots\binom{n_{q-1}}{\tau_{q-1}}(-Y_{1,l})^{n_1-\tau_1}\cdots(-Y_{q-1,l})^{n_{q-1}-\tau_{q-1}}. \tag{25}$$

On the other side,

$$\frac{1}{\omega_l(z)}=\prod_{s=1,s\neq l}^{p}\frac{1}{(z-\alpha_s)^{m_s}},$$

where

$$\frac{1}{(z-\alpha_s)^{m_s}}=\frac{(-1)^{m_s-1}}{(m_s-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{m_s-1}\frac{1}{z-\alpha_s}$$

and in a neighborhood of α_l

$$\frac{1}{z-\alpha_s}=\frac{1}{Z_{l,s}}\frac{1}{1-(Z_l/Z_{s,l})}=\frac{1}{Z_{l,s}}\sum_{v=0}^{\infty}\left(\frac{Z_l}{Z_{s,l}}\right)^v.$$

Therefore, by termwise differentiation we get

$$\frac{1}{(z-\alpha_s)^{m_s}}=\frac{1}{Z_{l,s}^{m_s}}\sum_{v_s=0}^{\infty}\binom{v_s+m_s-1}{v_s}\left(\frac{Z_l}{Z_{s,l}}\right)^{v_s}$$

and

$$\frac{1}{\omega_l(z)}=\left(\prod_{\substack{s=1\\ s\neq l}}^{p}\frac{1}{Z_{l,s}^{m_s}}\right)\prod_{\substack{s=1\\ s\neq l}}^{p}\left[\sum_{v_s=0}^{\infty}\binom{v_s+m_s-1}{v_s}\left(\frac{Z_l}{Z_{s,l}}\right)^{v_s}\right]=\sum_{h=0}^{\infty}\frac{Z_l^h}{h!}\cdot d_l^h\left(\frac{1}{\omega_l}\right),$$

where

$$d_l^h\left(\frac{1}{w_l}\right) = h!\left(\prod_{\substack{s=1\\ s\neq l}}^{p}\frac{1}{Z_{l,s}^{m_s}}\right)\sum_{\substack{(v_1,\dots,v_p)\in\mathbb{N}_0^p,\\ v_1+\cdots+v_p=h,\\ v_l=0}}\binom{v_1+m_1-1}{v_1}\cdots\binom{v_p+m_p-1}{v_p}\frac{1}{Z_{1,l}^{v_1}\cdots Z_{p,l}^{v_p}}. \tag{26}$$

Putting things together we finally get

$$\begin{aligned}\frac{1}{i!}d_l^i\left(\frac{Q}{\omega_l}\right) &= \left[\prod_{s=1,s\neq l}^{p}\frac{1}{Z_{l,s}^{m_s}}\right]\\ &\quad\times\sum_{j=0}^{i}\left(\sum_{\substack{(\tau_1,\dots,\tau_{q-1})\in\mathbb{N}_0^{q-1}\\ \tau_1+\cdots+\tau_{q-1}=j}}\binom{n_1}{\tau_1}\cdots\binom{n_{q-1}}{\tau_{q-1}}(-Y_{1,l})^{n_1-\tau_1}\cdots(-Y_{q-1,l})^{n_{q-1}-\tau_{q-1}}\right)\\ &\quad\times\sum_{\substack{(v_1,\dots,v_p)\in\mathbb{N}_0^p,\\ v_1+\cdots+v_p=i-j,\\ v_l=0}}\binom{v_1+m_1-1}{v_1}\cdots\binom{v_p+m_p-1}{v_p}\frac{1}{Z_{1,l}^{v_1}\cdots Z_{p,l}^{v_p}}.\end{aligned} \tag{27}$$

Next we calculate the numerator of the expression (24a). We have to expand the polynomial $\omega_l(x)\cdot P_{l,\lambda}(x)\cdot v_{l,\lambda}(x)$ at $x=\beta_m$. Using the shorthand notations

$$X_m := x-\beta_m \quad\text{and}\quad Y_{m,s} := \beta_m - \alpha_s,$$

we get

$$\begin{aligned}\omega_l(x)\cdot P_{l,\lambda}(x)\cdot v_{l,\lambda}(x) &= \left(\prod_{s=1,s\neq l}^{p}(x-\alpha_s)^{m_s}\right)\frac{1}{\lambda!}\sum_{i=0}^{m_l-\lambda-1}\frac{d_l^i(Q/\omega_l)}{i!}(x-\alpha_l)^{i+\lambda}\\ &= \left(\prod_{s=1,s\neq l}^{p}(X_m+Y_{m,s})^{m_s}\right)\frac{1}{\lambda!}\sum_{i=0}^{m_l-\lambda-1}\frac{d_l^i(Q/\omega_l)}{i!}(X_m+Y_{m,l})^{i+\lambda}.\end{aligned}$$

By making use of the binomial theorem and by interchanging the order of summation the last sum can be written

$$\sum_{\rho_l=0}^{m_l-1}X_m^{\rho_l}\left(\sum_{\rho=\max(0,\rho_l-\lambda)}^{m_l-\lambda-1}\frac{1}{\rho!}d_l^{\rho}\left(\frac{Q}{\omega_l}\right)\binom{\rho+\lambda}{\rho_l}Y_{m,l}^{\rho+\lambda-\rho_l}\right).$$

Therefore

$$\omega_l(x)\cdot P_{l,\lambda}(x)\cdot v_{l,\lambda}(x) = \sum_{j=0}^{n_m-1}\frac{X_m^j}{j!}D_m^j[\omega_l\cdot P_{l,\lambda}\cdot v_{l,\lambda}] + \mathcal{O}(X_m^{n_m}),$$

with

$$D_m^j[\omega_l\cdot P_{l,\lambda}\cdot v_{l,\lambda}] = \frac{j!}{\lambda!}\sum_{\substack{(\rho_1,\ldots,\rho_p)\in\mathbb{N}_0^p\\ \rho_1+\cdots+\rho_l=j}}\binom{m_1}{\rho_1}\cdots\binom{m_{l-1}}{\rho_{l-1}}\cdot Y_{m,1}^{m_1-\rho_1}\cdots Y_{m,l-1}^{m_{l-1}-\rho_{l-1}}$$
$$\times\left(\sum_{\rho=\max(0,\rho_l-\lambda)}^{m_l-\lambda-1}\frac{1}{\rho!}d_l^\rho\left(\frac{Q}{\omega_l}\right)\binom{\rho+\lambda}{\rho_l}Y_{m,l}^{\rho+\lambda-\rho_l}\right)$$
$$\times\binom{m_{l+1}}{\rho_{l+1}}\cdots\binom{m_p}{\rho_p}\cdot Y_{m,l+1}^{m_{l+1}-\rho_{l+1}}\cdots Y_{m,p}^{m_p-\rho_p}.$$

It remains to compute the derivatives occurring in (24b) in terms of the nodes and poles. We will do this by expanding $\omega_l(z)\cdot P_{l,\lambda}(z)\cdot v_{l,\lambda}(z)/(Q(z)\cdot z^{n_q-1})$ into a Laurent series around zero:

$$\frac{\omega_l(z)\cdot P_{l,\lambda}(z)\cdot v_{l,\lambda}(z)}{Q(z)\cdot z^{n_q-1}} = \sum_{\mu=1}^{n_q}A_{q,u}^{l,\lambda}\cdot\frac{1}{z^{n_q-\mu}}+\mathcal{O}\left(\frac{1}{z^{n_q}}\right)$$
$$=\sum_{j=0}^{n_q-1}A_{q,n_q-j}^{l,\lambda}\cdot\frac{1}{z^j}+\mathcal{O}\left(\frac{1}{z^{n_q}}\right),\quad z\to\infty.$$

We use the Laurent series of a typical factor of Q which is

$$\frac{1}{(z-\beta)^n}=\frac{1}{z^n}\sum_{v=0}^{\infty}\binom{v+n-1}{v}\left(\frac{\beta}{z}\right)^v \tag{28}$$

and Taylor's expansion around zero of the numerator which reads

$$\omega_l(z)\cdot P_{l,\lambda}(z)\cdot v_{l,\lambda}(z)=\frac{1}{\lambda!}\left(\prod_{s=1,s\neq l}^{p}(z-\alpha_s)^{m_s}\right)\sum_{\rho=0}^{m_l-\lambda-1}\frac{d_l^\rho(Q/\omega_l)}{\rho!}(z-\alpha_l)^{\rho+\lambda}$$

$$=\frac{1}{\lambda!}\left|\prod_{\substack{s=1\\ s\neq l}}^{p}\sum_{\rho_s=0}^{m_s}\binom{m_s}{\rho_s}z^{\rho_s}(-\alpha_s)^{m_s-\rho_s}\right|$$
$$\times\left(\sum_{\rho_l=0}^{m_l-1}z^{\rho_l}\cdot\sum_{\rho=\max\{0,\rho_l-\lambda\}}^{m_l-\lambda-1}\binom{\rho+\lambda}{\rho_l}(-\alpha_l)^{\rho+\lambda-\rho_l}\frac{d_l^\rho(Q/\omega_l)}{\rho!}\right)$$

$$=\frac{1}{\lambda!}\sum_{j=0}^{k-1}z^j\left[\sum_{\substack{(\rho_1,\ldots,\rho_p)\in\mathbb{N}_0^p\\ \rho_1+\cdots+\rho_p=j}}\binom{m_1}{\rho_1}\cdots\binom{m_{l-1}}{\rho_{l-1}}(-\alpha_1)^{m_1-\rho_1}\cdots(-\alpha_{l-1})^{m_{l-1}-\rho_{l-1}}\right.$$
$$\times\left(\sum_{\rho=\max\{0,\rho_l-\lambda\}}^{m_l-\lambda-1}\binom{\rho+\lambda}{\rho_l}(-\alpha_l)^{\rho+\lambda-\rho_l}\frac{d_l^\rho(Q/\omega_l)}{\rho!}\right)$$
$$\left.\times\binom{m_{l+1}}{\rho_{l+1}}\cdots\binom{m_p}{\rho_p}(-\alpha_{l+1})^{m_{l+1}-\rho_{l+1}}\cdots(-\alpha_p)^{m_p-\rho_p}\right].$$

By expanding each factor of Q according to (28) and by multiplying these expansions and the Taylor's expansion just derived we finally get

$$\frac{\omega_l(z)\cdot P_{l,\lambda}(z)\cdot v_{l,\lambda}(z)}{Q(z)\cdot z^{n_q-1}}=\sum_{j=0}^{n_q-1}\frac{1}{z^j}A^{l,\lambda}_{q,n_q-j}+\mathcal{O}\left(\frac{1}{z^{n_q}}\right),\quad z\to\infty,$$

with

$$A^{l,\lambda}_{q,n_q-j}=\frac{1}{\lambda!}\sum_{\substack{(v_1,\dots,v_q)\in\mathbb{N}_0^q\\ v_1+\cdots+v_q=j}}\binom{v_1+n_1-1}{v_1}\cdots\binom{v_{q-1}+n_{q-1}-1}{v_{q-1}}\beta_1^{v_1}\cdots\beta_{q-1}^{v_{q-1}}$$

$$\times\left\{\sum_{\substack{(\rho_1,\dots,\rho_p)\in\mathbb{N}_0^p\\ \rho_1+\cdots+\rho_p=k-1-v_q}}\binom{m_1}{\rho_1}\cdots\binom{m_{l-1}}{\rho_{l-1}}(-\alpha_1)^{m_1-\rho_1}\cdots(-\alpha_{l-1})^{m_{l-1}-\rho_{l-1}}\right.$$

$$\times\left(\sum_{\rho=\max\{0,\rho_l-\lambda\}}^{m_l-\lambda-1}\binom{\rho+\lambda}{\rho_l}(-\alpha_l)^{\rho+\lambda-\rho_l}\frac{d_l^\rho(Q/\omega_l)}{\rho!}\right)$$

$$\left.\times\binom{m_{l+1}}{\rho_{l+1}}\cdots\binom{m_p}{\rho_p}(-\alpha_{l+1})^{m_{l+1}-\rho_{l+1}}\cdots(-\alpha_p)^{m_p-\rho_p}\right\}\quad(j=0,\dots,n_q-1).$$

Now every thing we need is computed to give the final expressions which are purely in terms of the nodes and poles involved.

Theorem 3. *Under the assumptions of Theorem* 1 *we have for* $l=1,\dots,p$; $\lambda=0,\dots,m_l-1$:

$$A^{l,\lambda}_{m,\mu}=\begin{cases}\dfrac{\frac{1}{\lambda!}\sum_{\substack{(\rho_1,\dots,\rho_p)\in\mathbb{N}_0^p\\ \rho_1+\cdots+\rho_p=n_m-\mu}}\left(\prod_{\substack{s=1\\ s\neq l}}^p\binom{m_s}{\rho_s}Y_{m,s}^{m_s-\rho_s}\right)\cdot\sum_{\rho=\max\{0,\rho_l-\lambda\}}^{m_l-\lambda-1}\frac{1}{\rho!}d_l^\rho\left(\frac{Q}{\omega_l}\right)\binom{\rho+\lambda}{\rho_l}\cdot Y_{m,l}^{\rho+\lambda-\rho_l}}{\prod_{\substack{t=1\\ t\neq m}}^{q-1}(\beta_m-\beta_t)^{n_t}},\\[2ex] \quad m=1,\dots,q-1;\ \mu=1,\dots,n_m,\\[2ex] \dfrac{1}{\lambda!}\sum_{\substack{(v_1,\dots,v_q)\in\mathbb{N}_0^q\\ v_1+\cdots+v_q=n_q-\mu}}\left(\prod_{t=1}^{q-1}\binom{v_t+n_t-1}{v_t}\beta_t^{v_t}\right)\left\{\sum_{\substack{(\rho_1,\dots,\rho_p)\in\mathbb{N}_0^p\\ \rho_1+\cdots+\rho_p=k-1-v_q}}\left(\prod_{\substack{s=1\\ s\neq l}}^p\binom{m_s}{\rho_s}(-\alpha_s)^{m_s-\rho_s}\right)\right.\\[2ex] \quad\left.\times\left(\sum_{\rho=\max\{0,\rho_l-\lambda\}}^{m_l-\lambda-1}\binom{\rho+\lambda}{\rho_l}(-\alpha_l)^{\rho+\lambda-\rho_l}\frac{1}{\rho!}d_l^\rho\left(\frac{Q}{\omega_l}\right)\right)\right\},\quad m=q;\ \mu=1,\dots,n_q,\end{cases}$$

where $d_l^\rho(Q/\omega_l)$ *is given by* (27).

4. The adjoint and the inverse of a confluent Cauchy–Vandermonde matrix

In view of the general considerations at the beginning of Section 2 from Theorem 3 we obtain

Theorem 4. *Under the assumptions of Theorem* 1 *the adjoint of the Cauchy–Vandermonde matrix* (7) *equals*

$$V_{\mathrm{adj}} = (\det V)\cdot C^{\mathrm{T}},$$

where $\det V$ *can be found under* (14) *and the entries of the matrix* $C = (c_{j,i})_{j=1,\dots,k}^{i=1,\dots,k}$,

$$c_{j,i} := A_{\psi^{-1}(i)}^{\varphi^{-1}(j)}$$

are given in Theorem 3. *Here* φ^{-1} *and* ψ^{-1} *are the inverse functions of* (16) *and* (18), *respectively.*

Theorem 5. *Under the assumptions of Theorem* 1 *the inverse of the Cauchy–Vandermonde matrix* (7) *equals*

$$V^{-1} = C^{\mathrm{T}} = (c_{i,j})_{j=1,\dots,k}^{i=1,\dots,k},$$

where the matrix $C = (c_{j,i})_{j=1,\dots,k}^{i=1,\dots,k}$ *is defined in Theorem* 4.

5. Examples

Example 1. (Simple complex poles and simple nodes: Cauchy's interpolation problem). $(a_1,\dots,a_k) = (\alpha_1,\dots,\alpha_k)$, that means: $p = k$ and $m_i = 1$ for $i = 1,\dots,k$; $(b_1,\dots,b_k) = (\beta_1,\dots,\beta_k)$, that means $q-1 = k$ and $n_i = 1$ for $i = 1,\dots,k$, $n_q = 0$.

Then V is Cauchy's matrix

$$V = \begin{bmatrix} (\alpha_1-\beta_1)^{-1} & (\alpha_1-\beta_2)^{-1} & \cdots & (\alpha_1-\beta_k)^{-1} \\ (\alpha_2-\beta_1)^{-1} & (\alpha_2-\beta_2)^{-1} & \cdots & (\alpha_2-\beta_k)^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha_k-\beta_1)^{-1} & (\alpha_k-\beta_2)^{-1} & \cdots & (\alpha_k-\beta_k)^{-1} \end{bmatrix}.$$

By Theorem 4 its adjoint is

$$V_{\mathrm{adj}} = \det V\,(A_{m,1}^{l,0})_{m=1,\dots,k}^{l=1,\dots,k},$$

where

$$A_{m,1}^{l,0} = \prod_{\substack{s=1\\ s\neq l}}^{k} \frac{\beta_m-\alpha_s}{\alpha_l-\alpha_s}\,\frac{\prod_{t=1}^{k}(\alpha_l-\beta_t)}{\prod_{\substack{t=1\\ t\neq m}}^{k}(\beta_m-\beta_t)}$$

and where from (14) we infer

$$\det V = \prod_{\substack{i,j=1\\ i>j}}^{k} (\alpha_i - \alpha_j)(\beta_j - \beta_i) \Big/ \prod_{i,j=1}^{k} (\alpha_i - \beta_j).$$

By Theorem 5 the inverse of Cauchy's matrix (29) is

$$C^{\mathrm{T}} = (A^{l,0}_{m,1})^{l=1,\dots,k}_{m=1,\dots,k}.$$

The Lagrange basic functions in this case are

$$l_j(z) = \omega_j^0(z) = \prod_{s=1, s\neq j}^{k} \frac{z-\alpha_s}{\alpha_j - \alpha_s} \prod_{t=1}^{k} \frac{\alpha_j - \beta_t}{z - \beta_t} \quad (j = 1, \dots, k).$$

Example 2. (Taylor's interpolation at α_1 with a Cauchy–Vandermonde system).

$$(a_1, \dots, a_k) = (\alpha_1, \dots, \alpha_1), \quad \text{i.e. } p = 1 \text{ and } n_p = k;$$

$$(b_1, \dots, b_k) = (\underbrace{\beta_1, \dots, \beta_1}_{n_1}, \beta_2, \dots, \beta_{q-1}, \underbrace{\beta_q, \dots, \beta_q}_{n_q}) \quad \text{with } n_1 + \cdots + n_q = k,$$

$\beta_1, \dots, \beta_{q-1} \in \mathbb{C}$ pairwise distinct and distinct from $\alpha_1, \beta_q = \infty$ with $0 \leqslant n_q \leqslant k$.

The corresponding Cauchy–Vandermonde matrix V is

$$V = (d_1^\lambda u_{m,\mu})^{j=\psi(m,\mu)=1,\dots,k}_{\lambda=0,\dots,k-1},$$

where

$$d_1^\lambda u_{m,\mu} = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\lambda \frac{1}{(z-\beta_m)^\mu}\Big|_{z=\alpha_1} = \begin{cases} (-1)^\lambda \dfrac{(\mu+\lambda-1)!}{(\mu-1)!} \dfrac{1}{(\alpha_1 - \beta_m)^{\mu+\lambda}} \\ \quad \text{if } m = 1, \dots, q-1;\ \mu = 1, \dots, n_m, \\ \lambda! \dbinom{\mu-1}{\lambda} z^{\mu-1-\lambda} \\ \quad \text{if } m = q;\ \mu = 1, \dots, n_q. \end{cases}$$

By Theorem 4 its adjoint is

$$V_{\mathrm{adj}} = \det V\, (A^{1,\lambda}_{m,\mu})^{\lambda=0,\dots,k-1}_{j=\psi(m,\mu)=1,\dots,k},$$

where (cf. [4, p. 81])

$$\det V = \sigma_1 \left(\prod_{j=1}^{k} (j-1)! \right) \frac{\prod^{*k}_{\substack{i,j=1\\ i>j}} (b_i - b_j)}{\prod^{*k}_{j=1} (\alpha_1 - b_j)},$$

$\sigma_1 = (-1)^{\omega}$ with $\omega = \sum_{j=1}^{k} \mu_j(b_j)$, and where

$$A_{m,\mu}^{1,\lambda} =$$

$$\begin{cases} \dfrac{1}{\lambda!}\left[\displaystyle\sum_{\rho=\max\{0,n_m-\mu-\lambda\}}^{k-\lambda-1} \frac{1}{\rho!} d_1^{\rho} Q \binom{\rho+\lambda}{n_m-\mu} (\beta_m-\alpha_1)^{\rho+\lambda-n_m+\mu}\right] \Bigg/ \left[\prod_{\substack{t=1\\ t\neq m}}^{q-1} (\beta_m-\beta_t)^{n_t}\right] \\ \quad \text{if } m=1,\dots,q-1;\ \mu=1,\dots,n_m, \\ \dfrac{1}{\lambda!} \displaystyle\sum_{\substack{(v_1,\dots,v_q)\in\mathbb{N}_0^q\\ v_1+\cdots+v_q=n_q-\mu}} \left(\prod_{t=1}^{q-1}\binom{v_t+n_t-1}{v_t}\beta_t^{v_t}\right)\left[\sum_{\rho=\max\{0,k-1-v_q-\lambda\}}^{k-\lambda-1}\frac{1}{\rho!}d_1^{\rho}Q\binom{\rho+\lambda}{k-1-v_q}(-\alpha_1)^{\rho+\lambda-k+1+v_q}\right] \\ \quad \text{if } m=q;\ \mu=1,\dots,n_q, \end{cases}$$

with

$$\frac{d_1^{\rho}Q}{\rho!} = \sum_{\substack{(\tau_1,\dots,\tau_{q-1})\in\mathbb{N}_0^{q-1}\\ \tau_1+\cdots+\tau_{q-1}=\rho}} \left(\prod_{t=1}^{q-1}\binom{n_t}{\tau_t}(\alpha_1-\beta_t)^{n_t-\tau_t}\right).$$

By Theorem 5 the inverse of V is

$$C^{\mathrm{T}} = (A_{m,\mu}^{l,\lambda})_{j=\psi(m,\mu)=1,\dots,k}^{\lambda=0,\dots,k-1}.$$

The Lagrange basic functions in this case are

$$l_{\lambda}(z) = \omega_1^{\lambda}(z) = \frac{1}{\lambda!Q(z)} \sum_{i=0}^{k-\lambda-1} \frac{d_1^i Q}{i!}(z-\alpha_1)^{i+\lambda},$$

where $Q(z) = \prod_{j=1}^{*k}(z-b_j)$.

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